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Rolling tachyons from Liouville theory

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Abstract: In this work we propose an exact solution of the $c = 1$ Liouville model, i.e. of the world-sheet theory that describes the homogeneous decay of a closed string tachyon. Our expressions are obtained through careful extrapolation from the correlators of Liouville theory with $c \geq 25$. In the $c = 1$ limit, we find two different theories which differ by the signature of Liouville field. The euclidean limit coincides with the interacting $c = 1$ theory that was constructed by Runkel and Watts as a limit of unitary minimal models. The couplings for the lorentzian limit are new. In contrast to the behavior at $c > 1$, amplitudes in both $c = 1$ models are non-analytic in the momenta and consequently they are not related by Wick rotation.

Keywords: Bosonic Strings, Conformal Field Models in String Theory, Tachyon Condensation, Boundary Quantum Field Theory.

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1. Introduction

Over the last years we have made enormous progress in our understanding of the condensation processes that are triggered by closed and open string tachyons. Most of the results are based on the conjecture that the world-sheet models which realize the initial and final state of these condensation processes are related by renormalization group (RG) flows. This conjectured relation between space-time dynamics and world-sheet RG flows is still far from being understood, especially in the case of closed string tachyons (see e.g. [1, 2] for a review of recent progress and references). Many of the open issues can only be addressed through the construction of exact time dependent backgrounds. For the bounce of an open string tachyon such investigations were initiated by A. Sen [3, 4]. Studies of the closely related rolling tachyon solution mainly go back to the work of A. Strominger et al. [5, 6, 7]. This line of research has continued to provide important new insights into various aspects of open string tachyon condensation (see e.g. [8]–[13]), in particular for 2D string theory where quantum corrections to tree level results can be incorporated due to the duality with matrix models [14]–[17].

In most studies of such exact decay backgrounds, attention was restricted to special quantities in the world-sheet model, primarily to the boundary states, and very few attempts have been made to obtain the full solution, including the bulk and boundary 2- and 3-point couplings. This is the stage on which the results of this note are set. Our aim here is to obtain the full exact solution of the rolling closed string tachyon background. While closed string tachyons have certainly been a central issue, in particular for bosonic string theory, their condensation processes are rather difficult to picture, mainly because of the drastic effects they have on space-time itself. It therefore seems that the study of

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open string tachyons is a much better starting point. On the other hand, exact solutions of the boundary conformal field theories that appear in the description of branes and open strings are often facilitated if an appropriate basis of bulk states is chosen, associated with a corresponding bulk interaction. This is our main motivation to study the bulk theory first and we will see below how our results indeed represent a crucial step toward the solution of the boundary problem.

Before we state our results, let us recall that tachyon instabilities of a lorentzian $D+1$-dimensional static string background are related to relevant world-sheet fields $\Phi(z,\bar{z})$ of a unitary conformal field theory with D-dimensional euclidean target through

$$\delta S = \int_\Sigma dz d\bar{z}\, e^{iE_\Phi X_0}\, \Phi(z,\bar{z}) \qquad \text{where} \qquad E_\Phi^2 = \Delta_\Phi - 2\,.$$

Here, X_0 denotes the time-like free bosonic field that represents the time coordinate and $\Delta_\Phi = h_\Phi + \bar{h}_\Phi$ is the conformal weight of Φ. For relevant or marginally relevant fields, $\Delta_\Phi \leq 2$ implies a purely imaginary E_Φ so that δS represents an admissible perturbation of the original static background. The simplest example arises when $\Phi(z,\bar{z})$ is the identity field in which case δS does not couple the unitary spatial conformal field theory with the time component. Hence, we can focus on the perturbation of the time-like free boson X_0 with central charge $c = 1$. The whole setup resembles very much the problem of constructing Liouville theory with $c = 1$, only that now the signature of the Liouville direction X differs from the usual situation. As a consequence, the solution of the rolling tachyon background requires to obtain world-sheet correlators for the fields $\exp(2\alpha X) \sim \exp(-2i\alpha X_0)$ with real rather than imaginary parameter α.

What gives us hope to cope with this problem is the fact that Liouville has been solved for $c \geq 25$ in several steps throughout the last years [18]–[22]. Moreover, the solution is analytic, both in its dependence on the central charge and on the labels α. The latter fact implies that all couplings can be continued to imaginary α and hence it seems irrelevant that the Liouville direction is usually taken to be space-like. The analytic dependence on the central charge, on the other hand, may be used to continue the exact solution down to $c > 1$. But unfortunately, at the point $c = 1$, poles and zeros of the amplitudes move on top of each other, a behavior which has led to the widespread believe that the $c = 1$ limit is not well-defined. As we shall see below, this is not the case! In fact, theories with $c \leq 1$ can be extracted from a rigorous limit of the $c \geq 25$ solution, but the resulting models possess several unexpected and interesting features. As we shall argue, for $c \leq 1$ they do no longer depend smoothly on the central charge c. More importantly, their couplings are not analytic in the momenta α. This implies in particular that euclidean (E) and lorentzian (L) theories cannot be related by Wick rotation, i.e. by a continuation in the parameter α. Similar issues have also been discussed recently in [23], but in our case there exists a quite natural prescription to resolve the problem: If we want to go from the euclidean $c = 1$ theory to the lorentzian rolling tachyon solution, we first move the central charge back into the regime $c \geq 1$, continue analytically to real α and take then take the limit $c \to 1$. In this way we shall find two interacting $c = 1$ theories, one for imaginary and the other for real values of the parameter α.

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The 3-point couplings $C(\alpha_1, \alpha_2, \alpha_3)$ we propose for these two theories contain two different factors. One of them is analytic in the couplings α_j while the other is built from step functions. More precisely, we shall find that

$$\left\langle e^{2\alpha_1 X(z_1,\bar{z}_1)}\, e^{2\alpha_2 X(z_2,\bar{z}_2)}\, e^{2\alpha_3 X(z_3,\bar{z}_3)} \right\rangle = \frac{C^{\mathrm{S}}_{c=1}(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2h_{12}}|z_{23}|^{2h_{23}}|z_{13}|^{2h_{13}}}$$

where

$$z_{ij} = z_i - z_j\,, \quad h_{ij} = \alpha_i^2 + \alpha_j^2 - \alpha_k^2 \qquad \text{for} \qquad k \neq i,j$$

and

$$C^{\mathrm{S}}_{c=1}(\alpha_1,\alpha_2,\alpha_3) \;=\; (\pi\mu_{\mathrm{ren}})^{i\sum_j \alpha_j}\; e^{-2\pi\tilde{\alpha}}\, P^{\mathrm{S}}_{c=1}(\alpha_1,\alpha_2,\alpha_3) \times \\ \times \exp \int_0^\infty \frac{d\tau}{\tau}\,\frac{1}{\sinh^2\frac{\tau}{2}}\left[\sin^2\tilde{\alpha}\tau + \sum_{j=1}^{3}(\sin^2\tilde{\alpha}_j\tau - \sin^2\alpha_j\tau)\right]. \quad (1.1)$$

Here μ_{ren} was introduced as in [7] and the quantities $\tilde{\alpha}, \tilde{\alpha}_j$ are certain linear combinations of the parameters α_j to be defined in eqs. (2.4), (2.5). The superscript S stands for $\mathrm{S} = \mathrm{L}, \mathrm{E}$. As we explained above, the lorentzian theory $\mathrm{S} = \mathrm{L}$ requires $\alpha_j \in \mathbb{R}$ while in the euclidean case $\mathrm{S} = \mathrm{E}$ we must choose $\alpha_j = i\eta_j$ with $\eta_j \in \mathbb{R}$. The factor P^{S} is finally given by

$$P^{\mathrm{S}}_{c=1}(\alpha_1,\alpha_2,\alpha_3) = e^{4\pi\tilde{\alpha}}\left(1 + \Theta^{\mathrm{S}}(\tilde{\alpha})\prod_{j=1}^{3}\Theta^{\mathrm{S}}(\tilde{\alpha}_j)\right)$$

where $\Theta^{\mathrm{L}}(\alpha) = \Theta(\alpha)$ and $\Theta^{\mathrm{E}}(\alpha) = \theta(-i\alpha)$ are step functions Θ and θ on the real line, see eqs. (5.6) and (B.3). In writing this expression for P^{S} we have dropped some constant prefactors from the formulas below. Such factors can always be absorbed in the normalization of the fields. In the euclidean case, our formulas reproduce the couplings of an interacting $c = 1$ model that was constructed by Runkel and Watts in [24]. The lorentzian theory was previously considered by Strominger and Takayanagi [7]. A comparison with formula (4.17) of the latter paper shows that the two proposals differ mainly by the factor P^{L}. [1]

We shall begin our analysis with a short section on the minisuperspace model in which we review some of the results from [5] before computing a minisuperspace analogue of the 3-point coupling. The latter turns out to share some important features with the exact solution. In the third section we recall all necessary facts about the solution of Liouville theory with $c \geq 25$. The 3-point couplings will be rewritten in section 4 in a way that is well adapted to taking the central charge down to values $c \leq 1$. This enables us to make a precise proposal for the 3-point functions of $c \leq 1$ models. An explicit computation of our expressions boils down to the evaluation of a certain ratio of Jacobi ϑ-functions $\vartheta_1(x,\tau)$ at points $q = \exp \pi i\tau$ on the boundary $|q| = 1$ of the unit disc. For the $c = 1$ theory, we perform these calculations in the last section. This leads to the formulas we have spelled out above.

[1] It seems that the factor $\exp I$ (see eqs. (4.7),(4.8) of [7]) has been accidentally left out of the final proposal (4.17) of [7] for the 3-point coupling.

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2. The minisuperspace analysis

Before we start to descend into our analysis of the exact conformal field theories we would like to study a much simpler toy model that we will later recover as a 'minisuperspace limit' of the full solution. The analysis of the toy model was initiated by Strominger in [5]. Here, we shall review some elements of the model and then provide new expressions for an analogue of its 3-point function.

To gain some insights into the $c = 1$ Liouville theory, Strominger suggested to study solutions ϕ of the following Schroedinger equation with $\lambda > 0$,

$$H_0\,\phi := \left(\frac{\partial^2}{\partial x_0^2} + \lambda\, e^{2x_0}\right)\phi(x_0) - 4\omega^2\phi(x_0)\,. \tag{2.1}$$

The unusual sign in front of the derivative is associated with the fact that x_0 is thought of as a time-coordinate. As a consequence, the second term appears as if there was a potential that is unbounded from below. This certainly is reflected in the structure of the solutions which are given by arbitrary linear combinations of the function

$$\phi^{\rm in}_\omega(x_0) = \left(\frac{\lambda}{4}\right)^{i\omega}\Gamma(1-2i\omega)\,J_{-2i\omega}(\sqrt{\lambda}e^{x_0})\,. \tag{2.2}$$

and its complex conjugate. Here, J_ν is a Bessel function of the first kind. We have normalized the wave function such that in the far past

$$\phi^{\rm in}_\omega(x_0) \sim e^{-2i\omega x_0} \qquad \text{for} \qquad x_0 \to -\infty\,.$$

In the far future, on the other hand, the solution $\phi^{\rm in}_\omega$ takes the form

$$\phi^{\rm in}_\omega(x_0) \overset{x_0\to\infty}{\sim} \left(\frac{\lambda}{4}\right)^{i\omega-\frac{1}{4}}\Gamma(1-2i\omega)\,\frac{e^{-x_0/2}}{\sqrt{4\pi}}\,\left(e^{i\sqrt{\lambda}e^{x_0}-\frac{\pi}{4}(4\omega+i)} + e^{-i\sqrt{\lambda}e^{x_0}+\frac{\pi}{4}(4\omega+i)}\right).$$

Hence, positive and negative frequencies appear as we proceed in time. This can be considered as a signal for pair production [5] and we will explain how to read off the production rate below.

In [6] is was argued that the wave functions which correspond to fields in Liouville theory are special linear combinations of the wave function (2.2) and its complex conjugate. The reasoning is as follows: Liouville theory possesses an exponential potential with positive coefficients. Hence, among the two linear independent solutions there is one that decays exponentially while the other diverges at the same rate. The latter is clearly unphysical and so we end up with a single physical solution which, after Wick rotation, contains only positive frequencies. It corresponds to the function

$$\phi_\omega(x_0) = \left(\frac{\lambda}{4}\right)^{i\omega}\Gamma(1-2i\omega)\left(J_{-2i\omega}(\sqrt{\lambda}e^{x_0}) - e^{-2\pi\omega}\,J_{2i\omega}(\sqrt{\lambda}e^{x_0})\right)$$

with $\omega \geq 0$. An analogue of the 2-point function for these wave functions and its relation to the pair creation rate was discussed in [5].

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Here we are mainly interested in the 3-point function, since in the full conformal field theory this quantity encodes all the information about the exact solution. Its counterpart in the minisuperspace model can be evaluated through the following integral over a product of Bessel functions,

$$\langle\omega_1|e_{\omega_2}|\omega_3\rangle := \int_{-\infty}^{\infty} dx_0\, \phi_{\omega_1}(x_0)\, e^{-2i\omega_2 x_0}\, \phi_{\omega_3}(x_0) = (\lambda/4)^{2i\tilde{\omega}}\, P_0(\omega_j)\, e^{Q_0(\omega_j)}$$

where

$$\exp Q_0(\omega_1,\omega_2,\omega_3) = \frac{1}{\Gamma(1+2i\tilde{\omega})}\prod_{j=1}^{3}\frac{\Gamma(1+(-1)^j 2i\omega_j)}{\Gamma(1-(-1)^j 2i\tilde{\omega}_j)}\,, \tag{2.3}$$

$$P_0(\omega_1,\omega_2,\omega_3) = \frac{\pi i}{2\sinh 2\pi\tilde{\omega}} + \sum_{j=1}^{3}\frac{\pi i\, e^{(-1)^j 2\pi\tilde{\omega}_j - 2\pi\tilde{\omega}}}{2\sinh 2\pi\tilde{\omega}_j} = -\frac{\pi i\, e^{-2\pi\tilde{\omega}}}{2\sinh 2\pi\tilde{\omega}}\prod_{j=1}^{3}\frac{\sinh 2\pi\omega_j}{\sinh 2\pi\tilde{\omega}_j}\,.$$

The formula that was used to compute the integral can be found in standard mathematical tables. To write the answer more compactly, we have assigned the 4-tuple $\tilde{\omega}, \tilde{\omega}_j$ to the triple $\omega_j, j = 1,2,3$, by the simple prescription

$$\tilde{\omega} = \frac{1}{2}(\omega_1+\omega_2+\omega_3)\,, \qquad \tilde{\omega}_1 = \frac{1}{2}(\omega_2+\omega_3-\omega_1) \tag{2.4}$$

$$\tilde{\omega}_2 = \frac{1}{2}(\omega_3+\omega_1-\omega_2)\,, \qquad \tilde{\omega}_3 = \frac{1}{2}(\omega_1+\omega_2-\omega_3)\,. \tag{2.5}$$

Splitting the result of the integration in eq. (2.3) into the two factors P_0 and $\exp Q_0$ may seem a bit artificial at this point. But we shall see below that the split arises quite naturally when we recover the same expression through the minisuperspace limit of the exact theory.

From the 3-point function it is not hard to extract the 2-point function of our toy model in the limit $\omega_2 = \varepsilon \to 0$. Note that the factor P has poles at $\omega_1 - \omega_3 \pm \varepsilon = 0$ which, after taking the limit, produce the $\delta(\omega_1 - \omega_3)$ that we expect to find. More precisely, for $\omega_1, \omega_3 \geq 0$ we obtain

$$\lim_{\omega_2\to 0} C_0(\omega_1,\omega_2,\omega_3) \sim \delta(\omega_1-\omega_3)\, e^{-2\pi\omega_1}\left(\frac{\lambda}{4}\right)^{2i\omega_1}\frac{\Gamma(1-2i\omega_1)}{\Gamma(1+2i\omega_1)}\,.$$

The modulus of the quantity that multiplies the δ-function was interpreted in [5] as the pair production rate of this toy model.

3. Liouville theory with $c \geq 25$

The aim of this section is to review the solution of Liouville theory with $c \geq 25$. This solution was first proposed several years ago by H. Dorn and H.J. Otto [18] and by A. and Al. Zamolodchikov [19], based on extensive earlier work by many authors (see e.g. the reviews [25, 21] for references). Crossing symmetry of the conjectured 3-point function was then checked analytically in two steps by Ponsot and Teschner [20] and by Teschner [21, 22].

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As in any bulk conformal field theory, the exact solution of Liouville theory is entirely determined by the structure constants of the 3-point functions for the (normalizable) primary fields

$$\Phi_\alpha(z,\bar z) \sim e^{2\alpha X(z,\bar z)} \qquad \text{with} \qquad \alpha = \frac{Q_b}{2} + ip\,, \quad p \geq 0 \tag{3.1}$$

where $Q_b = b + b^{-1}, b \in \mathbb{R}$. These fields are primaries with conformal weight $h_\alpha = \bar h_\alpha = \alpha(Q_b - \alpha)$ under the action of the two Virasoro algebras whose central charge is $c = 1 + 6Q_b^2$. The couplings of three such fields are given by the following expression [18, 19].

$$C(\alpha_1,\alpha_2,\alpha_3) := \left[\pi\mu\gamma(b^2) b^{2-2b^2}\right]^{(Q_b-2\tilde\alpha)/b} \frac{\Upsilon'(0)}{\Upsilon(2\tilde\alpha - Q_b)} \prod_{j=1}^{3} \frac{\Upsilon(2\alpha_j)}{\Upsilon(2\tilde\alpha_j)} \tag{3.2}$$

where $\tilde\alpha$ and $\tilde\alpha_j$ are the linear combinations of α_j which are introduced just as in eqs. (2.4), (2.5) of the previous section, $\gamma(y) = \Gamma(y)/\Gamma(1-y)$ is a quotient of ordinary Γ-functions and the function $\Upsilon = \Upsilon_b$ is defined in terms of Barnes' double Γ-function Γ_2 (see appendix A) by

$$\Upsilon_b(\alpha) := \Gamma_2(\alpha|b,b^{-1})^{-1}\,\Gamma_2(Q_b - \alpha|b,b^{-1})^{-1}\,. \tag{3.3}$$

The properties of the double Γ-function which we spell out in appendix A imply that Υ possesses the following integral representation

$$\ln\Upsilon_b(y) = -2c_b + \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q_b}{2} - y\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q_b}{2} - y\right)\frac{t}{2}}{\sinh\frac{bt}{2}\,\sinh\frac{t}{2b}}\right]. \tag{3.4}$$

The constant c_b is defined in appendix A, but its value will not be relevant below. Moreover, we deduce from the two shift properties (A.2) of the double Γ-function that

$$\Upsilon_b(y+b) = \gamma(by)\, b^{1-2by}\,\Upsilon_b(y)\,, \quad \Upsilon_b(y+b^{-1}) = \gamma(b^{-1}y)\, b^{-1+2b^{-1}y}\,\Upsilon_b(y)\,. \tag{3.5}$$

Note that the second equation can be obtained from the first with the help of the self-duality property $\Upsilon_b(y) = \Upsilon_{b^{-1}}(y)$. We would like to stress that poles in Barnes double Γ-function give rise to zeros of the function Υ. The latter induce poles in the dependence of the coefficients (3.2) on the variables $\tilde\alpha_j$ and $\tilde\alpha$. This fact will become important below.

There are two further remarks that we would like to add. To begin with, we observe that the couplings (3.2) obey the following shift equation

$$\frac{C(\alpha_1+b,\alpha_2,\alpha_3)}{C(\alpha_1,\alpha_2,\alpha_3)} = -\frac{\gamma(-b^2)}{\pi\mu}\,\frac{\gamma(b(2\alpha_1+b))\gamma(b2\alpha_1)}{\gamma(b(2\tilde\alpha - Q_b))}\prod_{j=1}^{3}\frac{1}{\gamma(b2\tilde\alpha_j)} \tag{3.6}$$

and a dual equation with b being replaced by b^{-1}. This equation can be derived from the factorization constraint for the 4-point function of the three fields Φ_{α_j} with the fundamental Liouville field $\Phi = \Phi_{b/2}$. The derivation exploits the equation of motion of the Liouville field, see [26] for details. What we want to point out is that the two shift equation (3.6) and its dual counterpart possess a unique smooth (in b) family of solutions. Hence, the

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equations of motion along with the self-duality of Liouville theory fix the expression (3.2) for the 3-point function, given that the model exists for all $c \geq 25$ and that it depends smoothly on the central charge.

Let us finally stress that the expression (3.2) can be evaluated for values α_j other than those given in eq. (3.1). Even though the corresponding objects do not correspond to normalizable states of the model, they may be considered as well defined but non-normalizable fields. It is tempting to identify the identity field with limit $\lim_{\alpha \to 0} \Phi_\alpha$. This identification can indeed be confirmed by computing the corresponding limit of the coefficients C which is given by

$$\lim_{\alpha_2 \to 0} C(\alpha_1, \alpha_2, \alpha_3) = 2\pi\, \delta(\alpha_1 + \alpha_3 - Q_b) \, + \, R(\alpha_1)\, \delta(\alpha_1 - \alpha_3) \tag{3.7}$$

where

$$R(\alpha) = \big(\pi\mu\gamma(b^2)\big)^{(Q_b - 2\alpha)/b} \, \frac{b^{-2}\gamma(2b\alpha - b^2)}{\gamma(2 - 2b^{-1}\alpha + b^{-2})}$$

for all $\alpha_{1,3} = Q_b/2 + ip_{1,3}$ with $p_i \in \mathbb{R}$. If we restrict to the region where $p_1, p_3 \geq 0$, only the second term containing the *reflection amplitude* $R(\alpha)$ survives. The δ-functions again emerges from the singularities of C, just as in the minisuperspace example. We would like to stress that the reflection amplitude can also be obtained directly from the 3-point coupling without ever performing a limit $\alpha_2 \to 0$. In fact, in Liouville theory the fields with label α and $Q_b - \alpha$ differ only by a relative complex factor (see e.g. [21] for a more detailed discussion), namely by the factor $R(\alpha)$, i.e.

$$C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1)\ C(Q - \alpha_1, \alpha_2, \alpha_3)\,.$$

This remark shall become quite relevant below since it allows to recover the reflection amplitude without constructing the identity field first. Note that for the latter we had to continue correlation functions from the line $Q_b/2 + i\mathbb{R}$ into the complex plane.

4. On exact solutions for $c \leq 1$

We are now prepared to advance toward the solution of Liouville theory for $c \leq 1$. Our strategy is to continue the theory analytically from the regime $c \geq 25$ down to smaller central charges. This is achieved through analytic continuation of the parameter b into the complex plane. At first, such a continuation seems quite straightforward: as we move b along the unit circle, the central charge drops below $c = 25$ while the expression for the 3-point function (3.2) remains well defined. In fact, the construction of Barnes double Γ-function Γ_2 is not problematic unless $\mathrm{Re}(b) = 0$. Unfortunately, the values $c \leq 1$ of the central charge that we are interested in correspond precisely to values of b along the imaginary axis for which Barnes double Γ-function is not defined. Note, however, that the 2-point function (3.7), the shift equation (3.6) and its dual only involve ordinary Γ-functions and hence they possess a smooth continuation into the regime with $c \leq 1$.

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The last remark seems to suggest that we should simply look for a new solution of the shift equations (3.6) that we can use when $c \leq 1$. It turns out that such a solution is rather easy to write down. To this end, we introduce the following new function

$$Y_\beta(\alpha) := \Gamma_2(\beta + i\alpha|\beta, \beta^{-1})\, \Gamma_2(\beta^{-1} - i\alpha|\beta, \beta^{-1})\,. \tag{4.1}$$

This is defined for all $\beta = -ib$ with $\mathrm{Re}(\beta) \neq 0$. Again, using properties of Barnes double Γ-function we can deduce some important properties of the function Y. In particular, it possesses the following integral representation

$$\ln Y_\beta(\alpha) = 2c_\beta + \int_0^\infty \frac{d\tau}{\tau}\left[e^{-\tau}\left(\frac{Q_b}{2} - \alpha\right)^2 - \frac{\sin^2\left(\frac{Q_b}{2} - \alpha\right)\frac{\tau}{2}}{\sinh\frac{\beta\tau}{2}\sinh\frac{\tau}{2\beta}}\right] \tag{4.2}$$

with a constant c_β that is defined in appendix A. Similarly, we obtain shift properties of our new function Y from the behavior of Barnes double Γ-functions

$$Y(\alpha + i\beta) \;=\; \beta^{1-2i\alpha\beta}\,\gamma(i\alpha\beta)\, Y(\alpha)\,, \tag{4.3}$$
$$Y(\alpha - i\beta^{-1}) \;=\; \beta^{-1-2i\alpha\beta^{-1}}\,\gamma(-i\alpha/\beta)\, Y(\alpha)\,. \tag{4.4}$$

Observe in passing that the second equation is not obtained from the corresponding shift equation (3.5) by the formal substitution $b = i\beta$. While this difference will become important below it is not relevant for our construction of a solution to eqs. (3.6) and its dual. In fact, it is easy to obtain such a solution through the following combination of Y-functions,

$$\exp Q(\alpha_1, \alpha_2, \alpha_3) := \frac{Y(0)}{Y(2\tilde{\alpha} - Q)}\prod_{j=1}^{3}\frac{Y(2\alpha_j)}{Y(2\tilde{\alpha}_j)}\,. \tag{4.5}$$

Even though this is the unique smooth family of solutions to eq. (3.6) and its dual, we refrained from denoting it by C. Actually, we shall argue momentarily that the expression (4.5) cannot give the exact 3-point function. This then allows us to draw our first interesting conclusion: In the regime $c \leq 1$, Liouville theory does not depend smoothly on the central charge. Such a behavior is in sharp contrast to the properties of Liouville theory for $c > 1$.

There are two simple arguments that can be raised against an identification of the expression $\exp Q$ with the 3-point couplings. First of all, we want to stress that our function Y is defined as a product of Barnes Γ-functions Γ_2 and not through $1/\Gamma_2$ as in the construction (3.3) of the function Υ. Hence, the function Y has a discrete set of poles but no zeros. This in turn implies that the quantity $\exp Q$ has only zeros in the arguments $\tilde{\alpha}$. But as we pointed out before, poles in $\tilde{\alpha}$ are needed in order to recover the δ-function in the 2-point function upon taking α_2 to zero. Consequently, such a limit $\alpha_2 \to 0$ of $\exp Q$ cannot give a sensible 2-point function.

Our second argument relies on a comparison with the toy model in section 2. To this end, we choose the parameters α_j to be of the form

$$\alpha_1 = \frac{Q_b}{2} + \omega_1\beta\,, \qquad \alpha_3 = \frac{Q_b}{2} + \omega_3\beta\,, \qquad \alpha_2 = \omega_2\beta\,. \tag{4.6}$$

We can then take the limit $\beta \to 0$ of our expression for $\exp Q$ using the following asymptotic behavior of our function Y,

$$Y_\beta(\beta y) \sim Y_0 \beta^{iy}\, \Gamma(1+iy) + \cdots ,$$

along with the simple property $Y_\beta(Q_b - \alpha) = Y_\beta(\alpha)$. For the limit we then find that

$$\lim_{\beta\to 0} e^{Q(\alpha_1,\alpha_2,\alpha_3)} = \frac{1}{\Gamma(1+2i\tilde{\omega})} \prod_{j=1}^{3} \frac{\Gamma(1+(-1)^j 2i\omega_j)}{\Gamma(1-(-1)^j 2i\tilde{\omega}_j)} = e^{Q_0(\omega_1,\omega_2,\omega_3)} .$$

Hence, we are missing entirely the factor P_0 that was an important piece of 3-point function in our toy model. Recall that the poles in the factor P_0 were crucial in recovering the 2-point function from the 3-point coupling C_0.

In order to come up with a promising proposal for the 3-point function we will now try to understand how the function Υ_b and our new function Y_β are related to each other in the range of $b = i\beta$ for which they are both well defined. Let us stress that they cannot be the same, since their behavior under the two shifts differs by an exponential of an expression that is linear in α. This suggest that the quotient of Υ_b and Y_β can be expressed through Jacobi ϑ-functions. In fact it is possible to show that in the region where $\mathrm{Im}(b^2) = -\mathrm{Im}(\beta^2) > 0$, the two functions are related by

$$\Upsilon_{i\beta}(\alpha) = \mathcal{N}(\beta)\, e^{-\frac{\pi i}{2}(Q_b/2-\alpha)^2}\, e^{-\pi\frac{\alpha}{\beta}}\, \vartheta_1(i\alpha\beta^{-1}, \beta^{-2}) Y_\beta(\alpha) . \tag{4.7}$$

Here, $\mathcal{N}(\beta)$ is some factor that will not be relevant for the following and ϑ_1 is one of Jacobi's ϑ-functions (see appendix B for our conventions). One may test this formula simply by checking that both sides of the equality have the same behavior under the shifts $\alpha \to \alpha + b^{\pm 1}$. Another direct proof is spelled out in appendix A.

We are now prepared to spell out our proposal for the 3-point function of Liouville theory with $c \leq 1$. What we have shown so far is that

$$C(\alpha_1,\alpha_2,\alpha_3) = \Big[\pi\mu\gamma(-\beta^2)(-\beta^2)^{1+\beta^2}\Big]^{i(2\tilde{\alpha}-Q)/\beta} P(\alpha_1,\alpha_2,\alpha_3)\, e^{Q(\alpha_1,\alpha_2,\alpha_3)} \tag{4.8}$$

where

$$P(\alpha_i) = e^{-2\pi i Q_b\tilde{\alpha}} \frac{\mathcal{N}_0(\beta)\, \vartheta_1'(0)}{\vartheta_1(i(2\tilde{\alpha}-Q_b)\beta^{-1})} \prod_{j=1}^{3} \frac{\vartheta_1(i2\alpha_j\beta^{-1})}{\vartheta_1(i2\tilde{\alpha}_j\beta^{-1})} \tag{4.9}$$

and the function Q was defined in eq. (4.5). In the derivation we used that $\vartheta_1(x) = \vartheta_1(x,\beta^{-2})$ vanishes at $x=0$ and we absorbed a few β dependent constants into the overall normalization $\mathcal{N}_0$. By construction, the factor $\exp Q$ has a smooth continuation to the real β-axis. Our main claim is that (for appropriate choice of $\mathcal{N}_0(\beta)$) also P remains well defined for real β, at least after restricting the labels α_j to some subset of the complex plane. In general, however, P will not be an analytic function of the α_j but rather a distribution. We will compute this distribution for the $c=1$ theory below.

A very naive inspection of our formula for P could actually suggest that P does not exist at $\beta^2 = 1$. In fact, as we take β to the real axis, the modular parameter q is sent to the boundary of the unit disc and it is well known that Jacobi ϑ-functions are very singular for $|q| = 1$. At $\tau = \beta^{-2} = 1$, for example, $\vartheta_1(x)$ is a periodic δ-function in the real variable x (see e.g. [27]). Since products and quotients of a δ-function are not well defined, one might suspect that the same is true for P. Obviously, in this argument we have taken the limit too early. The correct prescription is to build P from the ϑ-functions first while β^2 is still within the lower half-plane and then to take the limit of the resulting family of functions.

We conclude this section with a short argument showing that the new factor P does resolve the mismatch between our earlier analysis of the minisuperspace limit of $\exp Q$ and the toy model from section 2. In fact, presenting ϑ_1 as an infinite sum it is easy to see that

$$\lim_{\beta\to 0} q^{-\frac{1}{4}}\vartheta_1(2i\omega, \beta^{-2}) = -2i \sinh 2\pi\omega\,.$$

Here, the point $\beta = 0$ is approached such that the imaginary part of β^2 is negative. Hence, $q = \exp i\pi/\beta^2$ tends to zero and so only the leading term in ϑ_1 can survive. Using this simple result, we can calculate the minisuperspace limit $\beta \to 0$ with labels (4.6) inserted into the couplings (4.8). The result agrees exactly with the couplings $C_0(\omega_1, \omega_2, \omega_3)$ of the toy model provided that we set $\lambda = 4\pi\mu$.

5. The $c = 1$ Liouville theory

We will finally approach the main goal of this note: in this section we shall obtain two possible $c = 1$ theories as a limit from Liouville theory. The first one is defined for purely imaginary labels α and we will identify it as the interacting $c = 1$ theory that was constructed by Runkel and Watts [24] as a limit of unitary minimal models. The second limit we take is defined for real α and it gives the 3-point couplings of the rolling tachyon solution. They agree with the proposal of Strominger and Takayanagi [7] only for a certain subset of parameters α_j.

Let us first try to analyse the $c = 1$ limit for $\alpha = i\eta$ purely imaginary. We claim that

$$P^{\mathrm{E}}_{c=1}(\eta_1, \eta_2, \eta_3) = \lim_{\beta\to 1} e^{2\pi Q_b\tilde\eta}\, \frac{\beta^{-1}Q_b\,\vartheta_1'(0)}{\vartheta_1((-2\tilde\eta - iQ_b)\beta^{-1})} \prod_{j=1}^{3} \frac{\vartheta_1(-2\eta_j\beta^{-1})}{\vartheta_1(-2\tilde\eta_j\beta^{-1})} \tag{5.1}$$

$$= \frac{\pi}{2} e^{4\pi i\tilde\eta} \left(\sum_{j=1}^{3} \theta(\tilde\eta_j) - \theta(\tilde\eta)\right)\left(1 + \theta(\tilde\eta)\prod_{j=1}^{3}\theta(\tilde\eta_j)\right). \tag{5.2}$$

Here, θ is the periodic step function with period length $L = 1$ (see appendix B) and the limit is taken such that $\mathrm{Im}(\beta^2) < 0$. This formula is proved using a number of identities between products of theta functions, starting from the following equation (see e.g. [27])

$$\vartheta_1(2x)\prod_j \vartheta_1(-2\eta_j\beta^{-1}) + \vartheta_3(2x)\prod_j \vartheta_3(-2\eta_j\beta^{-1}) =$$
$$= \vartheta_1(x - 2\tilde\eta\beta^{-1})\prod_j \vartheta_1(x + 2\tilde\eta_j\beta^{-1}) + \vartheta_3(x - 2\tilde\eta\beta^{-1})\prod_j \vartheta_3(x + 2\tilde\eta_j\beta^{-1})\,.$$

We differentiate this with respect to x and then set $x = 0$. Since $\vartheta_3'(0) = 0$, the second term on the left hand side drops out and we can use the resulting equality to replace the product of ϑ-functions in the numerator of P by a sum of two fourfold products with arguments being the same as in the denominator of P. The evaluation of the limit is thereby reduced to the evaluation of the limit for ϑ_1'/ϑ_1, ϑ_3'/ϑ_3 and ϑ_3/ϑ_1. These limits are computed in appendix B.

Our result demonstrates several of the general points we have made at the end of the previous section. Most importantly, the limit is completely well defined. But the resulting 3-point couplings are discontinuous in $\alpha = i\eta$ and hence there is no way to continue them naively beyond the real line. That is very different from the behavior of correlators at $c > 1$.

One may wonder whether we can really trust this outcome of our analysis. It could certainly happen that the limit is well defined but does not give consistent crossing symmetric couplings of a conformal field theory. But in the present situation, there is very strong additional evidence for the existence of this $c = 1$ model. In fact, some years ago, Runkel and Watts constructed an interacting non-rational $c = 1$ theory as a limit of unitary minimal models [24]. Their 3-point couplings also split into two factors. One of them possesses a nice integral representation, while the second jumps between 0 and 1. For this theory, Runkel and Watts tested the crossing symmetry numerically. We claim now that our couplings are related to those of the Runkel-Watts theory by a rather simple re-normalization of the fields.

The agreement of the smooth factor $\exp Q$ in the Runkel-Watts solution with the factor $\exp Q$ in our theory is obvious from the discussion in appendix A.1 of [24]. The relation between discontinuous factors can be seen if we rewrite our P^{E} in the following form

$$P^{\mathrm{E}}_{c=1} = \pi \prod_{j=1}^{3} e^{2\pi i \eta_j}\, \theta(2\eta_j) \cdot \frac{1}{4} \left(1 + \theta(\tilde{\eta}) \prod_{j=1}^{3} \theta(\tilde{\eta}_j) \right) . \tag{5.3}$$

Except for the first factors that can be absorbed in a change of normalization of the fields, this is the same as the function P defined in [24], though the detailed comparison is a bit messy since the authors of [24] did not provide a closed expression for this function.

We have emphasized at several places throughout this text how important poles in the 3-point function are to recover a sensible 2-point function in the limit $\alpha_2 \to 0$. But now we see that neither $\exp Q$ nor $P^{E}_{c=1}$ has poles at $\eta_1 - \eta_3 \pm \eta_2 = 0$. Fortunately, the problem is rather easy to understand: when we formulated such a requirement on the existence of poles before, we assumed that the identity field is simply the limit of Φ_α as $\alpha \to 0$. But this need not be the case. In fact, Runkel and Watts explained already how to cure the issue by constructing the identity field as a limit of the derivative $\partial_\eta \Phi_{i\eta}$. We refer the reader to [24] for details. The only new information that the 2-point function contains is the reflection amplitude which we can read off directly from the 3-point couplings (see discussion at the end of section 3). Using the invariance of $Y_{\beta=1}$ under reflection $\eta \to -\eta$ we obtain $R^{\mathrm{E}}_{c=1}(\eta) = -e^{2\pi i \eta}$. Let us stress, however, that the result includes a contribution from the factor that contains the coupling μ. To hide the divergence of $\gamma(-\beta^2)$ as $\beta \to 1$, one has to introduce a renormalized coupling μ_{ren}. The conventions we have used here are the same as in [7] and they are incorporated in our formula (1.1).

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Encouraged by the success of the euclidean limit we now turn to a second $c = 1$ limit that we take when the labels α are real. The analysis turns out to be a bit simpler than in the previous case. Using the modular properties of ϑ_1 we can show

$$P^{\mathrm{L}}_{c=1}(\alpha_1, \alpha_2, \alpha_3) = \lim_{\beta \to 1} e^{-2\pi i Q_b \tilde{\alpha}} \frac{\beta^{-1} Q_b \vartheta_1'(0)}{\vartheta_1(i(2\tilde{\alpha} - Q_b)\beta^{-1})} \prod_{j=1}^{3} \frac{\vartheta_1(i2\alpha_j \beta^{-1})}{\vartheta_1(i2\tilde{\alpha}_j \beta^{-1})} \tag{5.4}$$

$$= e^{4\pi\tilde{\alpha}} \lim_{\epsilon \to 0+} \frac{\pi}{\sin\pi(\frac{\tilde{\alpha}}{\epsilon} + i\tilde{\alpha})} \prod_{j=1}^{3} \frac{\sin\pi(\frac{\alpha_j}{\epsilon} + i\alpha_j)}{\sin\pi(\frac{\tilde{\alpha}_j}{\epsilon} + i\tilde{\alpha}_j)} \tag{5.5}$$

for $\alpha_j \in \mathbb{R}^+$. To compute the limit, we would like to rewrite the sin-factors in the denominator through a geometric series expansion. This requires to distinguish between four different regions, depending on the signs of $\tilde{\alpha}_j$. Since α_j are positive, the same is true for $\tilde{\alpha}$. But the signs of $\tilde{\alpha}_j$ vary. They can either be all positive, or it can happen that one of the $\tilde{\alpha}_j$ is negative. In each of these cases, an expansion can be performed and we are left with an infinite sum of exponentials. When we take the limit $\epsilon \to 0$, terms containing an $\exp i\alpha_j/\epsilon$ may be neglected since the 3-point function is considered as a distribution. Hence, we only have to look for the constants in our expansion. But such constant terms appear exclusively in the region where all $\tilde{\alpha}_j > 0$. This result can be spelled out more formally using the step function

$$\Theta(x) := \begin{cases} -1 & \text{for } x < 0 \\ +1 & \text{for } 0 < x\,. \end{cases} \tag{5.6}$$

From our above discussion we infer that for $\alpha_j > 0$

$$P^{\mathrm{L}}_{c=1}(\alpha_1, \alpha_2, \alpha_3) = \pi i e^{2\pi \sum_j \alpha_j} \left(1 + \Theta(\tilde{\alpha}) \prod_{j=1}^{3} \Theta(\tilde{\alpha}_j)\right). \tag{5.7}$$

Note that the form of the final expression for $P^{\mathrm{L}}_{c=1}$ is very similar to the corresponding formula (5.3) for its euclidean counterpart, only that the periodic step function θ has been replaced by Θ. In the region where all the $\tilde{\alpha}_j$ are positive, our result for P^{L} agrees with the findings of [7]. But the appearance of the step functions resolves a puzzle concerning the 2-point function of the model: As observed in [7], their formula for the 3-point function did not reproduce the right 2-point function upon sending $\alpha_2 \to 0$. In our result, the issue can be settled in the same way as it was settled for the Runkel-Watts theory. Once more, the reflection amplitude can be found without going into the details of how to construct the identity field. Analysing the behavior of $C^{\mathrm{L}}_{c=1}$ under reflections $\alpha_1 \to -\alpha_1$ we find $R^{\mathrm{L}}_{c=1}(\alpha) = \exp 2\pi\alpha$. As for the euclidean theory, this result depends on how the subtleties in the definition of μ_{ren} are dealt with. Here we have followed the conventions of [7].

6. Conclusion and outlook

In this note we have proposed an exact solution for Liouville theory with $c \leq 1$. The formulas were obtained through a rigorous limit from the solution of the model at $c \geq 25$.

Explicit expressions have been worked out for $c = 1$. In this case we found two different limiting theories with real conformal weights, depending on the range of the labels α. The euclidean limit coincides with an interacting $c = 1$ theory that was first constructed by Runkel and Watts as a limit of unitary minimal models. Our formulas for the lorentzian limit, however, are new and provide a very promising starting point for further studies of tachyon decays in string theory.

Even though understanding the homogeneous decay of bulk tachyons is certainly an important problem in (bosonic) string theory, the corresponding processes are expected to be rather violent. The condensation of open string tachyons are somewhat better behaved, mostly because such processes can be probed with closed strings. Finding expressions for 2- and 3-point functions of boundary Liouville theory, including the 2-point function between one bulk and one boundary field, is therefore of considerable interest. We have claimed in the introduction that the solution of the bulk problem presents an important step toward solving the boundary theory (with and without bulk coupling). Now we can see precisely how far we got. In fact, our analysis here was centered around understanding the function Υ which is built from a product of two double Γ-functions. But for the solution of the boundary theory we need to investigate one half of Υ, i.e. Barnes' double Γ-function itself (see e.g. [28]–[32] for some formulas concerning the models with $c \geq 25$). We shall address the relevant issues in a forthcoming publication [33]. Note that in the case of open strings, even the 2-point function is built from Barnes' double Γ-function and hence its $c = 1$-limit is expected to be quite non-trivial. A formula for this 2-point function was proposed recently in [6], but with our new techniques at hand it would be reassuring to confirm this proposal through a more rigorous derivation. Another interesting quantity that has not been studied yet is the 2-point function of a bulk- and a boundary field.

Let us finally stress that our analysis should also carry over to several other interesting decay processes. Most importantly, using the Sine-Liouville model (see e.g. [34] for an exact solution) it is possible to study the decay of a bulk tachyon with a sin-shaped profile in one space-like direction X. In the case of boundary perturbations, such profiles have received a lot of attention (see e.g. [35]–[39] for boundary conformal field theory treatments) because they allow to interpolate between Neumann and Dirichlet boundary conditions. One attractive feature of such models is that they possess a tunable parameter, namely the period length $L = 1/a$ of the profile. This allows to bring the relevant field $\Phi = \sin 2\pi a X$ arbitrarily close to marginality. It would be interesting to compare exact solutions of both the bulk and the boundary Sine-Liouville model for $c = 2$ with RG studies in the Sine-Gordon model, especially in the regime where the tachyon profile becomes marginal. We leave these problems to future investigations.

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A. Barnes' Γ-function, Υ and Y

We use this first appendix to collect some definitions and standard results concerning Barnes' double Γ-function (see also e.g. [40, 41]). These are used throughout the main text. We then spell out a direct proof of our relation (4.7).

Barnes' double Γ-function $\Gamma_a(y) = \Gamma_2(y|a, a^{-1})$ is defined for $y \in \mathbb{C}$ and complex a with $\mathrm{Re}(a) \neq 0$ such that its logarithm possesses the following integral representation,

$$\ln \Gamma_a(y) = c_a + \int_0^\infty \frac{d\tau}{\tau} \left[\frac{e^{-y\tau} - e^{-Q_a\tau/2}}{(1-e^{-a\tau})(1-e^{-\tau/a})} - \frac{\left(\frac{Q_a}{2} - y\right)^2}{2} e^{-\tau} - \frac{\frac{Q_a}{2} - y}{\tau} \right] \tag{A.1}$$

where $Q_a = a + a^{-1}$ and $c_a = \ln \Gamma_2(Q_a/2|a, a^{-1})$ is a constant. The integral exists for $0 < \mathrm{Re}(y)$. Under shifts by $a^{\pm 1}$ the function Γ_a behaves according to

$$\Gamma_a(y+a) = \sqrt{2\pi}\, \frac{a^{ay-\frac{1}{2}}}{\Gamma(ay)}\, \Gamma_a(y)\,, \qquad \Gamma_a(y+a^{-1}) = \sqrt{2\pi}\, \frac{a^{-\frac{y}{a}+\frac{1}{2}}}{\Gamma(a^{-1}y)}\, \Gamma_a(y)\,. \tag{A.2}$$

The functions Υ_b and Y_β defined through eqs. (3.3) and (4.1) in the main text are both constructed out of products of Barnes double Γ-function and it is easy to derive some of their main properties using the above formulas for Γ_a.

To prove the formula (4.7) we depart from the integral representation (3.4) and rotate the contour onto the positive imaginary axis,

$$\begin{aligned} \ln \Upsilon_b(\alpha) \;=\; & -2c_b + \int_0^\infty \frac{d\tau}{\tau} \left[e^{-i\tau} \left(\frac{Q_b}{2} - \alpha \right)^2 - \frac{\sin^2 \left(\frac{Q_b}{2} - \alpha \right) \frac{\tau}{2}}{\sinh \frac{\beta\tau}{2} \sinh \frac{\tau}{2\beta}} \right] + \\ & + 2i \sum_{n=1}^\infty \frac{(-1)^n}{n} \frac{\sin^2 \frac{\pi i n}{\beta} (Q_b/2 - \alpha)}{\sin \frac{\pi n}{\beta^2}} \end{aligned}$$

where $\beta = i\beta$, as usual. The second term is the contribution from the poles that we cross while rotating the contour. Note that, according to the integral formula (4.2), the first term can be expressed easily through our function Y_β. In addition, we rewrite the infinite sum by means of some simple trigonometric identities,

$$\ln \Upsilon_b(\alpha) \;\sim\; \ln Y_\beta(\alpha) - \frac{\pi i}{2} \left(\frac{Q_b}{2} - \alpha \right)^2 - i \sum_{n=1}^\infty \frac{1}{n} \sin \frac{2\pi n i \alpha}{\beta} + \frac{1}{n} \cot \frac{\pi n}{\beta^2} \cos \frac{2\pi n i \alpha}{\beta}$$

up to some constant depending only on b. It will drop out once we combine the Υ-function into a formula for the 3-point coupling. Our next step is to expand the factor $\cot \frac{\pi n}{\beta^2}$. In the region where $\mathrm{Im}(\beta^2) < 0$, we have

$$\cot \frac{\pi n}{\beta^2} = -i \left(1 + 2 \sum_{\nu=1}^\infty e^{\frac{2\pi i n \nu}{\beta^2}} \right).$$

We can now carry out the summation over n in the previous formula for $\ln \Upsilon$ to find that

$$\ln \Upsilon_b(\alpha) \sim \ln Y_\beta(\alpha) - \frac{\pi i}{2}\left(\frac{Q_b}{2} - \alpha\right)^2 + \ln\left(1 - e^{\frac{2\pi i\eta}{\beta}}\right)\prod_{\nu=1}^{\infty}\left(1 - q^{2\nu}e^{\frac{2\pi i\eta}{\beta}}\right)\left(1 - q^{2\nu}e^{\frac{-2\pi i\eta}{\beta}}\right)$$

where $q = \exp(\pi i/\beta^2)$. In the last term we recognize the product representation of a ϑ-function (see appendix 2) and hence we arrive at the formula (4.7).

B. Formulas for the $c = 1$ limit

In this technical appendix we state and prove several facts concerning the limit of various combinations of ϑ-functions as the modular parameter $\tau = 1/\beta^2$ is sent to the real line. Most of the properties of ϑ-functions that we will use can be found e.g. in [27]. Let us begin with the definition of the functions ϑ_1 and θ. In our conventions, the former is given by

$$\vartheta_1(x,\tau) := -2q^{\frac{1}{4}}\sin\pi x\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n}e^{2\pi i x})(1-q^{2n}e^{-2\pi i x}) \tag{B.1}$$

$$= \sum_{m=-\infty}^{\infty} q^{(m+\frac{1}{2})^2}\, e^{2\pi i(m+\frac{1}{2})(x+\frac{1}{2})} \tag{B.2}$$

where $q = \exp\pi i\tau$. We also use the Jacobi ϑ-function ϑ_3 which may be defined through the following sum

$$\vartheta_3(x,\tau) = \sum_{m=-\infty}^{\infty} q^{m^2}\, e^{2\pi i m x}\,.$$

Finally, we shall need two different step functions θ and θ_s, one of which is periodic, i.e. $\theta(x+1) = \theta(x)$, while the second behaves according to $\theta_s(x+1) = \theta_s(x) + 4$. On the interval $x \in [-1/2, 1/2]$ they both agree and are given by

$$\theta(x) = \theta_s(x) := \begin{cases} -1 & \text{for } -1/2 < x < 0 \\ +1 & \text{for } 0 < x < 1/2\,. \end{cases} \tag{B.3}$$

In our evaluation of the $c = 1$ limit of the 3-point function of Liouville theory we employ three simple formulas that we spell out now. The first of these formulas is

$$\lim_{\epsilon\to 0^+}\epsilon\frac{\vartheta_1'(x,1+i\epsilon)}{\vartheta_1(x,1+i\epsilon)} = -2\pi x + \pi\theta_s\left(\frac{1}{2}x\right) \qquad \text{for all} \quad x \in \mathbb{R}\,. \tag{B.4}$$

To compute the limit on the left hand side, we use the behavior of ϑ_1 under the modular transformations $\tau \to \tau - 1$ and $\tau \to -1/\tau$ and obtain

$$\begin{aligned}\lim_{\epsilon\to 0^+}\epsilon\frac{\vartheta_1'(x,1+i\epsilon)}{\vartheta_1(x,1+i\epsilon)} &= \lim_{\epsilon\to 0^+}\epsilon\frac{d}{dx}\ln\left(-\epsilon^{-\frac{1}{2}}e^{-\pi\frac{x^2}{\epsilon}}\vartheta_1\left(\frac{x}{i\epsilon},\frac{i}{\epsilon}\right)\right) \\ &= -2\pi x - i\lim_{\epsilon\to 0^+}\frac{\vartheta_1'(\frac{x}{i\epsilon},\frac{i}{\epsilon})}{\vartheta_1(\frac{x}{i\epsilon},\frac{i}{\epsilon})}\,.\end{aligned}$$

The second term is then evaluated by means of the expansion (see e.g. [42])

$$\frac{\vartheta_1'(y,\tau)}{\vartheta_1(y,\tau)} = \pi \cot \pi y + 4\pi \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin 2\pi n y$$

which we can use in the region $|\mathrm{Im}(y)| < \mathrm{Im}(\tau)$ where the sum is absolutely convergent. We can extend the resulting equality from the interval $x \in [-1,1]$ to the whole real line by using properties of the involved functions under shifts $x \to x+2$. This concludes the proof of eq. (B.4). Our second formula

$$\lim_{\epsilon \to 0^+} \epsilon \frac{\vartheta_3'(x, 1+i\epsilon)}{\vartheta_3(x, 1+i\epsilon)} = -2\pi x + \pi \theta_s \left(\frac{1}{2}x\right) \qquad \text{for all} \quad x \in \mathbb{R}. \tag{B.5}$$

is very similar to the first one and it is proved along the same lines. We leave this as an exercise. Finally, we also claim that

$$\lim_{\epsilon \to 0^+} \frac{\vartheta_3(x, 1+i\epsilon)}{\vartheta_1(x, 1+i\epsilon)} = i e^{-\frac{\pi i}{4}} \theta \left(\frac{1}{2}x\right) \qquad \text{for all} \quad x \in \mathbb{R}. \tag{B.6}$$

Modular properties of ϑ-functions are employed in a first step to see that

$$\lim_{\epsilon \to 0^+} \frac{\vartheta_3(x, 1+i\epsilon)}{\vartheta_1(x, 1+i\epsilon)} = e^{-\frac{\pi i}{4}} \lim_{\epsilon \to 0^+} \frac{\vartheta_1(\frac{x}{i\epsilon} + \frac{1}{2}, \frac{i}{\epsilon})}{\vartheta_1(\frac{x}{i\epsilon}, \frac{i}{\epsilon})}.$$

The limit is then evaluated with the help of the expansion (see e.g. [43])

$$\ln \frac{\vartheta_1(y+\frac{1}{2}, \tau)}{\vartheta_1(y,\tau)} = \ln \cot \pi y + 4 \sum_{m=1}^{\infty} \frac{1}{2m-1} \frac{q^{4m-2}}{1-q^{4m-2}} \cos \pi (4m-2) y$$

which holds for $|\mathrm{Im}(y)| < \mathrm{Im}(\tau)$. The extension to the full real line is obvious. The eqs. (B.4)–(B.6) provide the main ingredients in our computation of the Liouville 3-point function at $c=1$. This is explained in detail in section 5.

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